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Idea of Cube Root Expectation and Derivation of Its Definition

Dhritikesh Chakrabarty

Independent Researcher, Guwahati, Assam, India.

Abstract – Concept of cube root expectation has been introduced and defined based on cube root mean by the similar logic and the similar method used in defining arithmetic expectation, geometric expectation, harmonic expectation, quadratic expectation, and cubic expectation which had already been established in earlier studies. This article describes the concept and definition of cube root expectation.

Keywords: Random Variable, Cube Root Expectation, Concept, Definition.

1. INTRODUCTION

Mathematical expectation of a random variable was originally defined as the weighted average of the possible values assumed by the random variable with their respective probabilities as corresponding weights [1 , 3 , 17 , 19 , 21 , 28]. However, originally it had been defined with the help of weighted arithmetic mean and termed as mathematical expectation and later on it was termed as arithmetic expectation [9 , 11] due to the logic that it is based on arithmetic mean [4 , 6 - 8 , 24]. After that the concepts of geometric expectation [9 , 11], harmonic expectation [9 , 11], quadratic expectation [10 , 11] and cubic expectation [13] were introduced and defined on the basis of geometric mean [4 , 6 – 8 , 25], harmonic mean [4 , 6 – 8 , 26], quadratic mean [2 , 6 - 8 , 15 , 18 , 20 , 27] and cubic mean [6 - 8 , 14 , 22] respectively. Since expectation is defined using weighted measure average, it can also be possible to define expectation with the help of other weighted measures of average. Due to this reason, attempt has here been made on defining expectation using cube root mean [5 , 6 – 8 , 12 , 16]. Concept of cube root expectation has here been introduced and defined on the basis of cube root mean by applying the similar logic and the similar method applied in defining arithmetic expectation, geometric expectation, harmonic expectation, quadratic expectation and cubic expectation which had already been established in some earlier studies [9 - 11 , 13. This article describes the concept and definition of cube root expectation.

2. CUBE ROOT EXPECTATION OF A RANDOM VARIABLE

Let us consider a discrete finite random variable.

Suppose a real valued discrete random variable X assumes the values

$$x_1 , x_2 , \ldots\ldots\ldots , x_N ,$$

whose cube roots exist, so that N values constitute the population of the variable X.

Then the cube root mean [5 , 6 – 8 , 12 , 16] of X, denoted here by $\check{C}(X) = \check{C}(x_1 , x_2 , \ldots\ldots\ldots , x_N)$, is defined by

$$\check{C}(X) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^{\frac{1}{3}}\right)^3 \qquad (2.1)$$

This is the population cube root mean of X.

On the other hand, if the values of X correspond to the respective weights

$$w_1, w_2, \ldots\ldots, w_N$$

then the weighted cube root mean of X denoted here by

$$\check{C}(w : X) = \check{C}(w : x_1, x_2, \ldots\ldots, x_N),$$

will be

$$\check{C}(w : X) = \left(\frac{1}{w}\sum_{i=1}^{N} w_i x_i^{\frac{1}{3}}\right)^3 \qquad (2.2)$$

where $w = \sum_{i=1}^{N} w_i$

Now suppose, X assumes the values

$$x_1, x_2, \ldots\ldots, x_N$$

with respective probabilities

$$p_1, p_2, \ldots\ldots, p_N$$

Automatically, $\sum_{i=1}^{N} p_i = 1$

In this case, these probabilities can be regarded as the weights of the corresponding possible values assumed by the variable.

Accordingly, the weighted cube root mean of X in this case will be

$$\check{C}(p : X) = \check{C}(p : p_1, p_2, \ldots\ldots, p_N) = \left(\sum_{i=1}^{N} p_i x_i^{\frac{1}{3}}\right)^3 \qquad (2.3)$$

Since it describes the weighted cube root mean of all possible values of X with respective probabilities as the weights of the respective possible values assumed by the variable, it can be regarded as the cube root expectation of X.

Thus, cube root expectation can be defined as follows:

Definition (2.1):

If a real valued random variable X assumes the values $x_1, x_2, \ldots\ldots, x_N$, whose cube roots exist, with respective probabilities $p_1, p_2, \ldots\ldots, p_N$, then the cube root expectation of X, denoted by $E_{\check{c}}(X)$, is defined by

$$E_{\check{c}}(X) = \left(\sum_{i=1}^{N} p_i x_i^{\frac{1}{3}}\right)^3 \qquad (2.4)$$

Now, if X is a real valued discrete random variable assuming the countable infinite (denumerable) values

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$$x_1 , x_2 , \ldots\ldots$$

with respective probabilities

$$p_1 , p_2 , \ldots\ldots$$

then for the existence of cube root expectation of X, the term

$$\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}}$$

must exist.

Accordingly, cube root expectation can be defined, in this case, as follows:

Definition (2.2):

If X is a real valued discrete random variable assuming the countable infinite (denumerable) values x_1 , x_2 , ... with respective probabilities p_1 , p_2 , ... , then Eč(X) is defined by

$$E_{č}(X) = \left(\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}}\right)^3 \qquad (2.5)$$

provided $\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}}$ exists.

Now, suppose that X, instead of discrete, is a continuous random variable assuming values in the real valued interval

(a , b) or [a , b) or (a , b] or [a , b]

where each a & b of may be finite or infinite, with probability density function f(x).

Since the possible values in an interval is uncountable & infinite, the weighted cube root mean of these values is to be obtained by integration.

Accordingly, Eč(X) in this case is to be defined by

$$E_{č}(X) = \left\{\int_a^b x^{\frac{1}{3}} f(x)dx\right\}^3 \qquad (2.6)$$

Therefore, cube root expectation in this case of continuous random variable can be defined as follows:

Definition (2.3):

If X is a continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

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then Eč(X) can be defined by

$$\text{Eč}(X) = \{\int_a^b x^{\frac{1}{3}} f(x)dx\}^3 \qquad (2.7)$$

3. CUBE ROOT EXPECTATION OF FUNCTION OF A RANDOM VARIABLE

If φ(X) is a function of the real valued random variable X then proceeding with the same logic the cube root expectation of, denoted by Eč{φ(X)}, can be defined as follows:

Definition (3.1):

If a real valued discrete random variable X assumes the values $x_1, x_2, \ldots\ldots, x_N$ with respective probabilities $p_1, p_2, \ldots\ldots, p_N$, then Eč{$\varphi$(X)} can be defined by

$$\text{Eč}\{\varphi(X)\} = [\sum_{i=1}^{N} p_i\{\varphi(x_i)\}^{\frac{1}{3}}]^3 \qquad (3.1)$$

Definition (3.2):

If X is a real valued discrete random variable assuming countable infinite (denumerable) values $x_1, x_2, \ldots\ldots$ with respective probabilities $p_1, p_2, \ldots\ldots$, then the cube root expectation of a function φ(X) of X, denoted by Eč{φ(X)}, can be defined by

$$\text{Eč}\{\varphi(X)\} = [\sum_{i=1}^{\infty} p_i\{\varphi(x_i)\}^{\frac{1}{3}}]^3 \qquad (3.2)$$

provided $\sum_{i=1}^{\infty} p_i\{\varphi(x_i)\}^{\frac{1}{3}}$ exists.

Definition (3.3):

If X is a real valued continuous random variable assuming values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

the cube root expectation of a function φ(X) of X, denoted by Eč{φ(X)}, can be defined by

$$\text{Eč}\{\varphi(X)\} = [\int_a^b \{\varphi(x)\}^{\frac{1}{3}} f(x)dx]^3 \qquad (3.3)$$

4. RELATION BETWEEN CUBE ROOT AND ARITHMETIC EXPECTATIONS

Arithmetic expectation [9 , 11] of random variable X, denoted by EA(X), is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i \tag{4.1}$$

when X is finite

and by

$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i \tag{4.2}$$

when X is countable infinite (denumerable)

and also by

$$E_A(X) = \int_a^b x.f(x)dx \tag{4.3}$$

when X is continuous.

Similarly, the arithmetic expectation of a function φ (X) of X, denoted by EA{φ (X)}, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i \varphi(x_i) \tag{4.4}$$

when X is finite

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i \varphi(x_i) \tag{4.5}$$

when X is countable infinite (denumerable)

and also, by

$$E_A\{\varphi(X)\} = \int_a^b \varphi(x) f(x)dx \tag{4.6}$$

when X is continuous.

Putting

$$\varphi(X) = X^{\frac{1}{3}}$$

in any of the three definitions of $E_A\{\varphi(X)\}$and in any of the three definitions of $E_{\check{C}}\{\varphi(X)\}$, it is obtained that

$$E_{\check{C}}(X) = \{E_A(X^{\frac{1}{3}})\}3 \tag{4.7}$$

Thus, the cube root expectation of a random variable X is the cube of the arithmetic expectation of its cube root i.e. of $X^{\frac{1}{3}}$.

Similarly, putting

$$\varphi(X) = \{\psi(X)\}^{\frac{1}{3}}$$

in any of the three definitions of $E_A\{\varphi(X)\}$ and in any of the three definitions of $E_{\check{C}}\{\varphi(X)\}$, it is obtained that

$$E_{\check{C}}\{\psi(X)\} = E_A[\{\psi(X)\}^{\frac{1}{3}}]^3 \qquad (4.8)$$

which implies,

$$E_{\check{C}}\{\varphi(X)\} = E_A[\{\varphi(X)\}^{\frac{1}{3}}]^3 \qquad (4.9)$$

i.e. the cube root expectation of a function $\varphi(X)$ of X is the cube of the arithmetic expectation of the cube root of $\varphi(X)$ i.e. of $\{\varphi(X)\}^{\frac{1}{3}}$.

Thus, the following relationship, mentioned as theorem, between cube root expectation and arithmetic expectation has been obtained:

Theorem (4.1):

The cube root expectation of a random variable X is the cube of the arithmetic expectation of its cube root and also the cube root expectation of a function $\varphi(X)$ of a random variable X is the cube of the arithmetic expectation of the cube root of $\varphi(X)$.

5. EXAMPLE

Suppose,

a cube is to be selected at random from among the N cubes having the sides (in centimeter)

$$1, 2, 3, 4, \ldots\ldots, N$$

respectively

and X is a random variable which denotes the volume of the cube to be selected.

Then the possible values assumed by X are

$$1, 8, 27, 64, \ldots\ldots, N^3$$

with probability $\frac{1}{N}$ for each of them.

In this case,

$$E_{\check{c}}(X) = \{ \frac{1}{N} . \sqrt[3]{1} + \frac{1}{N} . \sqrt[3]{8} + \frac{1}{N} . \sqrt[3]{27} + \frac{1}{N} . \sqrt[3]{64} + ... + \frac{1}{N} . \sqrt[3]{N^3} \}3$$

$$= \{ \frac{1}{N} .(1 + 2 + 3 + 4 + ... + N)\}^3$$

$$= \{ \frac{1}{N} . \frac{N(N+1)}{2} \}^3$$

$$= \frac{(N+1)^3}{8}$$

6. CONCLUSION

The cube root mean of

$$x_1 , x_2 , ... , x_N$$

is a value μ such that the equation

$$\sqrt[3]{x_1} + \sqrt[3]{x_2} + ... + \sqrt[3]{x_N} = \sqrt[3]{\mu} + \sqrt[3]{\mu} + ... + \sqrt[3]{\mu}$$

is satisfied [8]. Thus, the cube root mean is a measure of average which is relevant to analysing data related to volume, 3D structures, and physical processes etc. where values change proportionally to the cube root of a dimension [5 , 16 , 22]. Accordingly, cube root expectation is relevant to the random variable which change following cube root law. In this connection it is to be mentioned that cubic mean is also relevant to analysing data related to volume, 3D structures, and physical processes etc. where values changes proportionally to the cube of a dimension [5 , 16 , 22] which implies cubic expectation is relevant to the random variable which changes following cube root. Thus, the cubic expectation and the cube root expectation are both relevant to analysing data related to volume, 3D structures, and physical processes etc. but the pattern of change of variable in the former one is reverse to that in the later one.

One important point to be noted is that concept of expectation plays significant role in the theory of statistical estimation specifically in defining unbiasedness of estimator [23 , 29]. Accordingly, the concept of cube root expectation is also significant in this area of statistical inference.

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