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Cubic Unbiased Estimator: Some Properties

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Abstract - Unbiasedness of estimator, which had originally been defined with the help of arithmetic expectation and was defined later with the help of geometric expectation, harmonic expectation & quadratic expectation, has here been defined based on cubic expectation and termed as cubic unbiased estimator. Concept of cubic unbiased estimator, with some of its properties, has been discussed in this article.

Keywords: Estimation, Cubic Unbiasedness, CUE, Properties.

1. INTRODUCTION

In the literature of statistic, unbiasedness is a property or quality of estimator which is desirable though not essential [1 , 10 , 11 , 12]. The concept of unbiasedness had originally been introduced/defined with the help of the concept of mathematical expectation or equivalently arithmetic expectation [1 , 2 , 10 – 12 , 14 - 17] of estimator and was termed arithmetic unbiased estimator [3] or simply unbiased estimator. Later on, it was introduced/defined with the help of the concept of geometric expectation, harmonic expectation & quadratic expectation and termed respectively as geometric unbiased estimator, harmonic unbiased estimator & quadratic unbiased estimator along with identifying some properties of them [3 - 7]. In this attempt, unbiasedness of estimator has been introduced/defined with the help of the concept of cubic expectation [8 , 9]. Attempt has also been made on identifying some properties satisfied by it.

2. CUBIC UNBIASED ESTIMATOR OF PARAMETER

Let us consider a population following a probability function having parameter θ.

Suppose

$$\{X_1 , X_2 , \ldots\ldots\ldots , X_n\}$$

is a random sample drawn from this population and

$$T = T(X_1 , X_2 , \ldots\ldots\ldots , X_n)$$

is an estimator of the parameter θ.

As per the concept of arithmetic unbiasedness, T is arithmetic unbiased estimator of θ if and only if

$$E_A(T) = \theta$$

where $E_A(T)$ is the arithmetic expectation of T.

However, if

$$E_A(T) \neq \theta$$

then the term $B_A(T)$ given by

$$B_A(T) = E_A(T) - \theta$$

is the arithmetic bias of estimator T of θ.

Accordingly, T is arithmetic unbiased estimator of θ if and only if

$$B_A(T) = 0$$

i.e. if and only if the arithmetic bias of T is 0.

Now, Applying the concept of cubic expectation [8 , 9] in the concept of arithmetic unbiased estimator as mentioned above, it is obtained that

T can be regarded as cubic unbiased estimator of parameter θ if and only if

$$E_C(T) = \theta$$

where $E_C(T)$ is the cubic expectation of T.

On the other hand, if

$$E_C(T) \neq \theta$$

then the term $B_C(T)$ given by

$$B_C(T) = E_C(T) - \theta$$

can be regarded as the cubic bias of estimator T of θ.

Accordingly, T is cubic unbiased estimator of θ if and only if

$$B_C(T) = 0$$

i.e. if and only if the cubic bias of T is 0.

3. CUBIC UNBIASED ESTIMATOR OF FUNCTION OF PARAMETER

The above concepts/definitions of cubic unbiased can be generalized to the case of function of parameter.

Let

$$S = S(X_1 , X_2 , \ldots\ldots\ldots , X_n)$$

be an estimator of $\phi(\theta)$, a function of parameter θ.

Then S can be regarded as cubic unbiased estimator of $\phi(\theta)$ if and only if

$$E_C(S) = \phi(\theta)$$

In the case where

$$E_C(S) \neq \phi(\theta) ,$$

the term BC(S) given by

$$B_C(S) = E_C(S) - \phi(\theta)$$

can be regarded as the cubic bias of estimator S of $\phi(\theta)$.

Accordingly, S is cubic unbiased estimator of $\phi(\theta)$ if and only if

$$B_C(S) = 0$$

i.e. if and only if the cubic bias of S is 0.

4. SOME PROPERTIES

Let us abbreviate cubic unbiased estimator by CUE and arithmetic unbiased estimator by AUE.

Property (1): If T is CUE of θ then T^3 is AUE of θ 3 and conversely if T is AUE of θ the $T^{\frac{1}{3}}$ is CUE of $\theta^{\frac{1}{3}}$.

Proof: Cubic expectation of a random variable X is the absolute cube root of arithmetic expectation of its cube [8 , 9] i.e.

$$E_C(X) = \{E_A(X^3)\}^{\frac{1}{3}}$$

This implies,

$$E_C(X^{\frac{1}{3}}) = \{E_A(X)\}^{\frac{1}{3}}$$

Thus,

$$E_C(T) = \{E_A(T^3)\}^{\frac{1}{3}} \quad \& \quad E_C(T^{\frac{1}{3}}) = \{E_A(T)\}^{\frac{1}{3}}$$

The first one of these two implies,

$$\text{if } E_C(T) = \theta \text{ then } E_C(T\ 3) = \theta\ 3$$

i.e. if T is CUE of θ the T^3 is AUE of θ 3.

Similarly the second one of these two implies,

$$\text{if } E_A(T) = \theta \text{ then } E_C(T^{\frac{1}{3}}) = \theta^{\frac{1}{3}}$$

i.e. if T is AUE of θ the $T^{\frac{1}{3}}$ is CUE of $\theta^{\frac{1}{3}}$.

Property (1): If T is CUE of parameter θ then cT is CUE of parameter cθ where c is non-zero real constant.

Proof: EQ(cT) can be expressed as

$$E_Q(cT) = [E_A\{(cT)2\}]^{\frac{1}{2}} = \{c2E_A(T\ 2)\}^{\frac{1}{2}} = c\{E_A(T\ 2)\}^{\frac{1}{2}} = c.E_Q(T) = c\theta$$

Therefore, cT is CUE of parameter cθ.

Corollary: If T is CUE of parameter θ then -T is CUE of parameter -θ.

This follows from the fact that

$$E_Q(-T)\ E_Q(-1.T) = = -1.E_Q(T) = -\theta$$

Property (2): Cubic mean (abbreviated as CM) of a finite number of CUEs of a parameter θ is also CUE of θ.

Proof: Suppose, T and S are two CUEs of θ.

Equation

$$E_C(X) = \{E_A(X^3)\}^{\frac{1}{3}}$$

implies that

$$E_C\,(X^{\frac{1}{3}}) = \{E_A(X)\}^{\frac{1}{3}}$$

Thus,

$$E_C\,(\text{CM of T and S}) = E_C\,[\{\tfrac{1}{2}(T3\ +S3)\}^{\frac{1}{3}}] = [E_A\{\tfrac{1}{2}(T3\ +S3\}]^{\frac{1}{3}}$$

$$= [\tfrac{1}{2}\{E_A(T3) + E_A(S3)\}]^{\frac{1}{3}}$$

But

$$E_A(T3) = \{E_C(T)\}3 = \theta\,3 \quad \& \quad E_A(S3) = \{E_C(S)\}3 = \theta\,3$$

Therefore,

$$E_C\,(\text{CM of T and S}) = \{\tfrac{1}{3}\,(\theta\,3 + \theta\,3)\}^{\frac{1}{3}} = \theta$$

Therefore, CM of T and S is CUE of θ.

Now suppose,

$$T_1\ ,T_2\ ,\ \ldots\ldots\ldots\ ,\ T_k$$

are CUEs of θ.

Then

$$E_Q\,(\text{CM of } T_1\ ,T_2\ ,\ \ldots\ldots\ldots\ ,\ T_k)$$

$$= E_C[\{\tfrac{1}{k}\,(T_1 3 + T_2 3 + \ldots\ldots\ldots + T_k 3)\}^{\frac{1}{3}}]$$

$$= [E_A\{\tfrac{1}{k}\,(T_1 3 + T_2 3 + \ldots\ldots\ldots + T_k 3)\ \}]^{\frac{1}{3}}$$

$$= [\tfrac{1}{k}\{E_A(T_1 3) + E_A(T_2 3) + \ldots\ldots\ldots + E_A(T_k 3)\}]^{\frac{1}{3}}$$

But,

$$E_A(T_i 3) = E_C(T_1)\}3 = \theta\,3\ ,\ (i = 1,\ 2,\ \ldots\ldots\ldots, k)$$

Therefore,

$$E_Q\,(\text{CM of } T_1\ ,T_2\ ,\ \ldots\ldots\ldots\ ,\ T_k) = \{\tfrac{1}{k}\,(\theta\,3 + \theta\,3 + \ldots\ldots\ldots + \theta\,3)\}^{\frac{1}{3}} = \theta$$

Therefore, CM of $T_1\ ,T_2\ ,\ \ldots\ldots\ldots\ ,\ T_k$ is CUE of θ.

Property (3): There may exists more than one CUE of a parameter.

Proof: Let us consider a population following the discrete uniform distribution [13] described by the probability mass function

$$P(X = x_i) = \frac{1}{K}\ ,\ (x_i = 1, 2, \ldots\ldots\ldots, K)$$

with μC as its population CM where

$$\mu C = (13 .\tfrac{1}{K} + 23 .\tfrac{1}{K} + \ldots\ldots\ldots + K\,3 .\tfrac{1}{K})^{\frac{1}{3}}$$

Suppose,

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$$\{X_1, X_2, \ldots\ldots, X_n\}$$

is a random sample drawn from this population.

Then each element of the sample assumes the values

$$1, 2, \ldots\ldots, K$$

with equal probability $\frac{1}{K}$,

so that by the definition of cubic expectation,

$$E_C(X_i) = (1^3 \cdot \tfrac{1}{K} + 23 \cdot \tfrac{1}{K} + \ldots\ldots + K^3 \cdot \tfrac{1}{K})^{\frac{1}{3}} = \mu C \quad ,$$

for each X_i (i = 1, 2, ……, n).

This implies each X_i is a CUE of μC .

By Property (2),

CM of any two elements of the sample is CUE of μC ,

Similarly, CM of any three elements of the sample is also CUE of μC ,

CM of any four elements of the sample is also CUE of μC

and so on.

Thus, a parameter may have more than one CUE of itself.

Property (4): There may not exists CUE of a parameter.

Proof: In the case of discrete uniform distribution, mentioned in Property (3), if we consider the population arithmetic mean μA given by

$$\mu A = (1 \cdot \tfrac{1}{K} + 2 \cdot \tfrac{1}{K} + \ldots\ldots + K \cdot \tfrac{1}{K})$$

then CUE does not exist for this parameter μA .

Property (5): If T is CUE of parameter θ and S is CUE of parameter φ then CM of T and S is CUE of the CM of θ and φ.

In general, if

$$T_1, T_2, \ldots\ldots\ldots, T_k$$

are CUEs of the respective parameters

$\theta_1, \theta_2, \ldots\ldots\ldots, \theta_k.$,

then the CM of $T_1, T_2, \ldots\ldots, T_k$, is CUE of the CM of $\theta_1, \theta_2, \ldots\ldots, \theta_k$.

Proof: We have

$E_C(\text{CM of T and S}) = E_C\,[\{\frac{1}{2}(T_3 + S^3)\}^{\frac{1}{3}}] = [E^A\{\frac{1}{2}(T^3 + S^3\}]^{\frac{1}{3}}$

$= [\frac{1}{2}\{E_A(T^3) + E_A(S^3)\}]^{\frac{1}{2}}$

But

$$E_A(T3) = \{E_C(T)\}3 = \theta\,3 \;\;\&\;\; E_A(S^3) = \{E_C(S)\}3 = \varphi 3$$

Therefore,

$$E_C(\text{CM of T and S}) = \{\frac{1}{2}(\theta 3 + \varphi\,3)\}^{\frac{1}{3}} = \text{CM of } \theta \text{ and } \varphi$$

Therefore, CM of T and S is CUE of CM of θ and φ.

Again,

$$T_1\;, T_2\;,\;\ldots\ldots\ldots\;,\;T_k$$

are CUEs of the respective parameters

$$\theta_1\;, \theta_2\;,\;\ldots\ldots\ldots\;, \theta_k.\,,$$

This implies,

$E_A(T_i 3) = E_C(T_i)\}3 = \theta_i 3$, (i = 1 , 2 , ……… , k)

This implies,

E_C (CM of T_1 , T_2 , ……… , T_k)

$= E_C\,[\{\frac{1}{k}(T_1 3 + T_2 3 + \ldots\ldots\ldots + T_r 3)\}^{\frac{1}{3}}]$

$= [E_A\{\frac{1}{k}(T_1 3 + T_2 3 + \ldots\ldots\ldots + T_k 3)\}]^{\frac{1}{3}}$

$= [\frac{1}{k}\{E_A(T_1 3) + EA(T_2 3) + \ldots\ldots\ldots + EA(T_r 3)\}]^{\frac{1}{3}}$

$= \{\frac{1}{k}(\theta_1 3 \;+ \theta_2 3 \quad + \ldots\ldots + \theta_k 3)\}^{\frac{1}{3}}$

= CM of θ_1 , θ_2 , ……… , θ_r

Therefore,

$$\text{CM of } T_1\;, T_2\;,\;\ldots\ldots\ldots\;,\;T_k$$

is CUE of the

$$\text{CM of } \theta_1\;, \theta_2\;,\;\ldots\ldots\ldots\;, \theta_k$$

5. CONCLUSION

The concept and definition of cubic unbiasedness is based on cubic expectation which is based on cubic mean and therefore can be useful and helpful in determining unbiased estimator of parameter which is of cubic nature. Properties of cubic unbiased estimator, identified here, are can also be helpful in finding unbiased estimator of a function of parameter of similar type. Moreover, the above properties of cubic unbiased estimator can contribute to searching for more properties of cubic unbiased estimator and enriching the theoretical aspect of statistical estimation.

REFERENCES

[1] Birnbaum Allan (1961): "A Unified Theory of Estimation, The Annals of Mathematical Statistics. 32(1), 112 – 135. doi:10.1214/aoms/1177705145.

[2] Chattamvelli, R., Shanmugam, R. (2024). "Mathematical Expectation", In: Random Variables for Scientists and Engineers. Synthesis Lectures on Engineering, Science, and Technology. Springer, Cham. https://doi.org/10.1007/978-3-031-58931-7_1.

[3] Dhritikesh Chakrabarty (2025): "Arithmetic, Geometric, Harmonic and Quadratic Unbiasedness in Estimation", Partners Universal International Innovation Journal, 3(4), 24 – 32. www.puiij.com. DOI: https://doi.org/10.5281/zenodo.17060844.

[4] Dhritikesh Chakrabarty (2025): "Continuous Random Variable Arithmetic, Geometric, Harmonic and Quadratic Expectations", Partners Universal Innovative Research Publication, 03(04), 36 – 42. www.puirp.com. DOI:10.5281/zenodo.16996732.

[5] Dhritikesh Chakrabarty (2025): "Geometric Unbiased Estimator: Some Properties", Partners Universal International Innovation Journal, 03(05), 69 – 75. www.puiij.com. DOI:10.5281/zenodo.17444460.

[6] Dhritikesh Chakrabarty (2025): "Harmonic Unbiased Estimator: Some Properties", Partners Universal Innovative Research Publication, 03(05), 92 – 96. www.puirp.com. DOI:10.5281/zenodo.17536909.

[7] Dhritikesh Chakrabarty (2025): "Quadratic Unbiased Estimator: Some Properties", Partners Universal Innovative Research Publication (PUIRP), (ISSN: 3048-586X), 03(06), 136 – 142. www.puirp.com. DOI:10.5281/zenodo.18037586.

[8] Dhritikesh Chakrabarty (2026): "Idea of Cubic Expectation and Derivation of Its Definition", Partners Universal Innovative Research Publication, 04(01), 136 – 142. www.puirp.com. DOI:10.5281/zenodo.18776915.

[9] Dhritikesh Chakrabarty (2026): "Additive Property of Cubic Expectation", Partners Universal Innovative Research Publication, 04(02), 92 – 97. www.puirp.com. DOI:10.5281/zenodo.20027876.

[10] Junting Liu (2025): "Analysis of Existence and Validity of Unbiased Estimation", Theoretical and Natural Science, 125(1), 193 – 201. DOI: 10.54254/2753-8818/2025.GL26548.

[11] Lehmann E. L. (1951): "A General Concept of Unbiasedness", The Annals of Mathematical Statistics, 22(4), 587 – 592. doi:10.1214/aoms/1177729549. JSTOR 2236928.

[12] Nikulin M. and Voinov V. (2025): "Unbiased Estimators and Their Applications", In book: International Encyclopedia of Statistical Science. DOI: 10.1007/978-3-662-69359-9_733.

[13] Pepelyshev A. & Zhigljavsky A. (2020): "Discrete Uniform and Binomial Distributions with Infinite Support", Soft Comput, 24, 17517 – 17524. https://doi.org/10.1007/s00500-020-05190-2 .

[14] Pfeiffer P. E. (1990): "Mathematical Expectation", In: Probability for Applications. Springer Texts in Statistics, Springer, New York, NY. https://doi.org/10.1007/978-1-4615-7676-1_15.

[15] Taylor Courtney (2019): "Unbiased and Biased Estimators", Thought Co, Retrieved 2020-09-12.

[16] Voinov Vassily G.; Nikulin, Mikhail S. (1993): "Unbiased Estimators and Their Applications", Vol. 1: Univariate case. Dordrect: Kluwer Academic Publishers. ISBN 0-7923-2382-3.

[17] Yadav S. K., Singh S. & Gupta, R. (2019): "Random Variable and Mathematical Expectation", In: Biomedical Statistics, Springer, Singapore. https://doi.org/10.1007/978-981-32-9294-9_26.