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Additive Property of Cubic Expectation

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Abstract – The concept of cubic expectation had been introduced in an earlier study, and its definition was formulated in the case of random variables (both discrete and continuous). The additive property of cubic expectation has here been derived in the case of a random variable and in the case of a function of a random variable. Derivation of the property has been presented in this article.

Keywords: Random Variable, Quadratic expectation, Additive Property.

1. INTRODUCTION

The theory of mathematical expectation, associated to random variable, had primarily been developed during the period from the middle of the 17th century to the mid-nineteenth with the works mainly by the mathematicians Blaise Pascal, Chevalier de Méré, Pierre de Fermat, Christiaan Huygen, Pierre-Simon Laplace, Pafnuty Chebyshev and W. A. Whitworth [1 , 8 – 11 , 13, 19 , 20]. Originally, mathematical expectation of a random variable had been defined as weighted average of its all possible values with their respective probabilities as corresponding weights [2 , 14 , 16 , 17] which was later termed as arithmetic expectation [3]. As continuation of the development of the theory of expectation, three more concepts of measure of expectation had been introduced and consequently formulated their definitions. These geometric expectation [3, 4 , 6], harmonic expectation [3 , 4 , 6] and quadratic expectation [5 , 6]. In a recent study, one more concept of expectation namely cubic expectation [7] has been introduced along with formulating its definition in the case of random variables (both discrete and continuous). However, it is yet to study on the properties of cubic expectation. Here, attempt has been made to search for if cubic expectation possesses some property of additive nature. The property obtained has been presented in this article.

2. CUBIC EXPECTATION

Let X be a real valued random variable and φ(X) be a function X.

If X is finite discrete and assumes the values

$$x_1 , x_2 , ... , x_N$$

with respective probabilities

$$p_1 , p_2 , ... , p_N ,$$

then the cubic expectation of X, denoted by E_C(X), is defined by

$$E_C(X) = (\sum_{i=1}^{N} p_i x_{i3})^{1/3} \qquad (2.1)$$

while the cubic expectation of φ(X), denoted by EC{φ(X)}, is defined by

$$E_C\{\varphi(X)\} = [(\textstyle\sum_{i=1}^{N} p_i.\{\varphi(x_i)\}^3]^{1/3} \qquad (2.2)$$

If X is denumerable (i.e. countable infinite) discrete and assumes the values

$$x_1, x_2, \ldots\ldots\ldots\ldots$$

with respective probabilities

$$p_1, p_2, \ldots\ldots\ldots\ldots$$

then $E_C(X)$ is defined by

$$E_C(X) = (\textstyle\sum_{i=1}^{\infty} p_i x_{i3})^{1/3} \qquad (2.3)$$

provided $\sum_{i=1}^{\infty} p_i x_{i3}$ exists

while $E_C\{\varphi(X)\}$ is defined by

$$E_C\{\varphi(X)\} = [(\textstyle\sum_{i=1}^{\infty} p_i.\{\varphi(x_i)\}3]^{1/3} \qquad (2.4)$$

provided $\sum_{i=1}^{\infty} p_i.\{\varphi(x_i)\}^3$ exists.

Again, if X is a continuous and assumes values in any of the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then $E_C(X)$ is defined by

$$E_C(X) = \{\int_a^b x^3 f(x)dx\}^{\frac{1}{3}} \qquad (2.5)$$

while $E_C\{\varphi(X)\}$ is defined by

$$E_C\{\varphi(X)\} = [\int_a^b \{\varphi(x_i)\}^3 f(x_i)]^{\frac{1}{3}} \qquad (2.6)$$

Note (2.1):

Arithmetic expectation of X, denoted by $E_A(X)$, is defined by

$$E_A(X) = \textstyle\sum_{i=1}^{N} p_i x_i \qquad (2.7)$$

when X is finite discrete

and by

$$E_A(X) = \textstyle\sum_{i=1}^{\infty} p_i x_i \qquad (2.8)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i x_i$ exists

and by

$$E_A(X) = \int_a^b x.f(x)dx \quad (2.9)$$

when X is continuous [3 , 4 , 6].

Similarly, the arithmetic expectation of φ (X), denoted by $E_A\{\varphi(X)\}$, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i . \varphi(x_i) \quad (2.10)$$

when X is finite discrete

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i . \varphi(x_i) \quad (2.11)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i . \varphi(x_i)$ exists

and by

$$E_A\{\varphi(X)\} = \int_a^b \varphi(x) f(x)dx \quad (2.12)$$

when X is continuous [3 , 4 , 6].

3. ADDITIVE PROPERTY IN THE CASE OF RANDOM VARIABLE

Let X & Y be two real valued random variables

such that

$E_A(X)$, $E_A(Y)$, $E_C(X)$ & $E_C(Y)$

exist.

Then from the definitions of $E_A(X)$ and $E_C(X)$,

$\{E_C(X)\}3 = E_A(X\ 3)$ & $\{E_C(Y)\}3 = E_A(Y^3)$

i.e. $E_C(X) = \{E_A(X)\}^{1/3}$ & $E_C(Y) = \{E_A(Y\ 3)\}^{1/3}$

Replacing X by $X^{1/3}$ and Y by $Y^{1/3}$, it is obtained that

$E_C(X^{1/3}) = \{E_A(X)\}^{1/3}$ & $E_C(Y^{1/3}) = \{E_A(Y)\}^{1/3}$

Accordingly,

$$E_C\{(X + Y)1/3\} = \{E_A(X + Y)\}^{1/3}$$

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Additive property of arithmetic expectation [12 , 15 , 18] implies that

$$E_A(Y + Y) = E_A(Y) + E_A(Y)$$

which implies,

$$[E_C \{(X + Y)1/3\}]3 = \{E_C (X\, 1/3)\}3 + \{E_C (Y\, 1/3)\}3 \qquad (3.1)$$

This can be regarded as additive property of cubic expectation of two random variables.

In general, if

$$X_1 , X_2 , \ldots\ldots , X_k$$

are k random variables,

then proceeding with the same logic, it can be obtained that

$$[E_C \{(X_1 + X_2 + \ldots\ldots + X_k)1/3\}]3 = \{E_C (X_1 1/3)\}3 + \{E_C (X_2 1/3)\}3 + \ldots\ldots$$

$$\ldots + \{E_C (X_k 1/3)\}3 \qquad (3.2)$$

This can be regarded as general additive property, i.e. additive property of finite number of random variables, of cubic expectation.

4. ADDITIVE PROPERTY IN THE CASE OF FUNCTION OF RANDOM VARIABLE

Let X & Y be two real valued random variables and

$$\phi(X) = \phi X \quad \& \quad \psi(Y)\} = \psi Y$$

be two functions of X & Y respectively such that

$$E_C (X) , E_C (Y) , E_C(\phi X) , E_C(\psi Y)$$

exist.

Then from the definitions of $E_A\{\varphi(X)\}$ and $EC\{\varphi(X)\}$

$$\{E_C (\phi X)\}3 = E_A(\phi X^3) \quad \& \quad \{E_C (\psi Y)\}3 = EA(\psi Y^3)$$

$$\text{i.e.} \quad E_C (\phi X) = \{E_A (\phi X)\}^{1/3} \quad \& \quad E_C (\psi Y) = \{E_A(\psi Y^3)\}^{1/3}$$

Replacing ϕX by $\phi X1/3$ and ψY by $\psi Y^{1/3}$, it is obtained that

$$E_C (\phi X^{1/3}) = \{E_A (\phi X)\}^{1/3} \quad \& \quad E_C (\psi Y^{1/3}) = \{E_A (\psi Y)\}^{1/3}$$

This implies,

$$E_C \{(\phi X + \psi Y)1/3\} = \{E_A (\phi X + \psi Y)\}^{1/3}$$

Additive property of arithmetic expectation [12 , 15 , 18] implies that

$$E_A(\phi X + \psi Y) = E_A(\phi X) + E_A(\psi Y)$$

Thus,

$$[E_C \{(\phi X + \psi Y)1/3\}]3 = \{E_C (\phi X\, 1/3)\}^3 + \{E_C (\psi Y\, 1/3)\}^3 \qquad (4.1)$$

This can be regarded as additive property of cubic expectation in the case of finite number of functions of random variables.

In general, if

$$X_1, X_2, \ldots\ldots\ldots, X_k$$

are k random variables

and

$$\phi 1(X_1) = \phi 1,\ \phi 2(X_1) = \phi 2,\ \ldots\ldots\ldots,\ \phi k(X_k) = \phi k$$

are real valued functions of them respectively,

then proceeding with the same logic, it can be obtained that

$$[E_C\{(\phi 1 + \phi 2 + \ldots\ldots\ldots + \phi k)1/3\}]^3 = \{E_C(\phi 11/3)\}^3 + \{E_C(\phi 21/3)\}^3 + \ldots\ldots\ldots$$
$$\ldots\ldots + \{E_C(\phi k^{1/3})\}^3 \qquad (4.2)$$

This can be regarded as additive property of cubic expectation in the case of finite number of functions of random variables.

5. CONCLUSION

Additive property of cubic expectation obtained above can be summarized as in the following theorem:

Theorem:

Cube of cubic expectation of the cube root of the sum of a finite number of random variables is the sum of cubes of individual cubic expectations of the cube roots of the variables provided all these cubic expectations exist.

and

Cube of cubic expectation of cube root of the sum of a finite number of functions of single random variables is the sum of cubes of individual cubic expectations of the cube roots of the functions provided all these cubic expectations exist.

At this stage, it is to be mentioned that cubic expectation may carry more unknown properties. Accordingly, this may be a problem of further research. Researcher may invest effort to search for the hidden properties of cubic expectation. Similarly, searching for relationship between cubic expectation and each of arithmetic expectation, geometric expectation, harmonic expectation, and quadratic expectation may be other problem(s) of further research.

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