Idea of Cubic Expectation and Derivation of Its Definition

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Abstract – In continuation to the concepts of arithmetic expectation, geometric expectation, harmonic expectation and quadratic expectation, concept of cubic expectation has been introduced in this article and defined in a similar manner on the basis of cubic mean. This article describes the concept and definition of cubic expectation.

Keywords: Random Variable, Cubic Expectation, Concept, Definition.

1. INTRODUCTION

The concept of expectation associated to a random variable, which was originated in the middle of the 17th century, was introduced as weighted average of the possible values assumed by the random variable with their respective probabilities as corresponding weights [1 , 27]. At first, it was defined as the weighted arithmetic mean of the possible values assumed by the random variable with their respective probabilities as corresponding weights and termed as mathematical expectation [4 , 18 , 22 , 26] later termed as arithmetic expectation [7] since it is based on the oldest and widely used measure of average namely arithmetic mean [5 , 23]. Later on, three other concepts of measure of expectation had been introduced on the basis of geometric mean [5 , 24], harmonic mean [5 , 25] and quadratic mean (or root mean square) [3 , 15 , 17 , 26] and consequently respective definitions were formulated which were respectively termed as geometric expectation [7 , 9], harmonic expectation [7 , 9] and quadratic expectation [8 , 9]. In this connection, it is to be mentioned that it can also be possible to derive a measure of expectation with the help of other measures of average. That is why the same has here been attempted on the basis of cubic mean [12, 21]. In continuation to the concepts of arithmetic expectation, geometric expectation, harmonic expectation and quadratic expectation, concept of cubic expectation has been introduced in this article and defined in a similar manner on the basis of cubic mean. This article describes the concept and definition of cubic expectation.

2. CUBIC EXPECTATION OF A RANDOM VARIABLE

Let us consider a discrete finite random variable.

Suppose a real valued discrete random variable X assumes the values

$$\mathrm{x}_1 \, , x_2 \, , \ldots\ldots\ldots \, , x_N$$

so that N values constitute the universe/population of the variable X.

Then the cubic mean [12 , 21] of X, denoted by C(X) = C($\mathrm{x}_1 , x_2 , \ldots\ldots\ldots , x_N$), is defined by

$$\mathrm{C(X)} = \{\tfrac{1}{N} \, (\mathrm{x}_1 3 + x_2 3 + \ldots\ldots\ldots + x_N 3 \,)\}^{\frac{1}{3}} = (\tfrac{1}{N} \textstyle\sum_{i=1}^{N} \; x_N 3 \,)^{\frac{1}{3}}$$

This is the population cubic mean or the universe cubic mean of X.

On the other hand, if the values of X correspond to the respective weights

$$w_1 , w_2 , \ldots\ldots\ldots , w_N$$

then, C(X) = C($x_1 , x_2 , \ldots\ldots\ldots , x_N$), will be weighted cubic mean of X and is accordingly defined by

$$C(X) = \left(\frac{1}{\sum_{i=1}^{N} w_i} \sum_{i=1}^{N} w_i x_i 3 \right)^{\frac{1}{3}}$$

Now suppose, X assumes the values

$$x_1 , x_2 , \ldots\ldots\ldots , x_N$$

with respective probabilities

$$p_1 , p_2 , \ldots\ldots\ldots , p_N$$

Automatically, $\sum_{i=1}^{N} p_i = 1$

Then C(X), in this case, will be the weighted cubic mean of X with the probabilities as the respective weights of its respective possible values and accordingly is defined by

$$C(X) = \left(\sum_{i=1}^{N} p_i x_i 3 \right)^{\frac{1}{3}}$$

Since it describes the weighted cubic mean of all possible values of X with respective probabilities as the weights of the respective values, it can be regarded as the cubic expectation of X. Thus, cubic expectation can be defined as follows:

Definition (2.1):

If a real valued random variable X assumes the values $x_1 , x_2 , \ldots\ldots\ldots , x_N$ with respective probabilities $p_1 , p_2 , \ldots\ldots\ldots , p_N$, then the cubic expectation of X, denoted by EC(X), is defined by

$$E_C(X) = \left(\sum_{i=1}^{N} p_i x_i 3 \right)^{\frac{1}{3}} \qquad (2.1)$$

Now, if X is a real valued discrete random variable assuming the countable infinite (denumerable) values

$$x_1 , x_2 , \ldots\ldots\ldots\ldots\ldots$$

with respective probabilities

$$p_1 , p_2 , \ldots\ldots\ldots\ldots\ldots$$

then for the existence of cubic expectation of X, the term

$$\sum_{i=1}^{\infty} p_i x_i 3$$

must converge to a finite number.

Definition (2.2):

If X is a real valued discrete random variable assuming the countable infinite (denumerable) values $x_1 , x_2 , \ldots\ldots\ldots\ldots\ldots$ with respective probabilities $p_1 , p_2 , \ldots\ldots\ldots\ldots\ldots$, then the cubic expectation of X, denoted by E_C(X), is defined by

$$E_C(X) = \left(\sum_{i=1}^{\infty} p_i x_i 3 \right)^{\frac{1}{3}} \qquad (2.2)$$

provided $\sum_{i=1}^{\infty} p_i x_i 3$ is convergent and is a finite number.

Now, suppose that X, instead of discrete, is a continuous random variable assuming values in the real valued interval (a , b) or [a , b) or (a , b] or [a , b]

where each a & b of may be finite or infinite,

with probability density function f(x).

Since the possible values in an interval is uncountable & infinite, the weighted cubic mean of these values is to be obtained by integration.

Accordingly, $E_C(X)$ in this case is to be defined by

$$E_C(X) = \{\int_a^b x^3 f(x)dx\}^{\frac{1}{3}}$$

Definition (2.3):

If X is a continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then cubic expectation of *X*, denoted by $E_C(X)$, can be defined by

$$E_C(X) = \{\int_a^b x^3 f(x)dx\}^{\frac{1}{3}} \qquad (2.3)$$

3. CUBIC EXPECTATION OF FUNCTION OF A RANDOM VARIABLE

If φ(X) is a function of the real valued random variable X then proceeding with the same logic the cubic expectation of, denoted by EC{φ(X)}, can be defined as follows:

Definition (3.1):

If a real valued random variable X assumes the values x_1 , x_2 , ………. , x_N with respective probabilities p_1 , p_2 , ………. , p_N , then the cubic expectation of a function φ(X) of X, denoted by $E_C\{\varphi(X)\}$, can be defined by

$$E_C\{\varphi(X)\} = [\ \sum_{i=1}^{N}\ p_i\{\phi(x_i)\}^3]^{\frac{1}{3}} \qquad (3.1)$$

Definition (3.2):

If X is a real valued discrete random variable assuming countable infinite (denumerable) values x_1 , x_2 , ……………. with respective probabilities p_1 , p_2 , ……………. , then the cubic expectation of a function φ(X) of X, denoted by $E_C\{\varphi(X)\}$, can be defined by

$$E_C\{\varphi(X)\} = [\ \sum_{i=1}^{\infty}\ p_i\{\phi(x_i)\}^3]^{\frac{1}{3}} \qquad (3.2)$$

provided $\sum_{i=1}^{\infty}\ p_i\{\phi(x_i)\}^3$ is convergent and is a finite number.

Definition (3.3):

If X is a continuous random variable assuming real values in the intervals

$$(a\,,b)\ \text{ or }\ [a\,,b)\ \text{ or }\ (a\,,b]\ \text{ or }\ [a\,,b]$$

where a b may be finite or infinite,

having probability density function f(x),

the cubic expectation of a function φ (X) of X, denoted by $E_C\{\varphi(X)\}$, can be defined by

$$E_C\{\varphi(X)\} = [\int_a^b \{\varphi(x_i)\}^3 f(x_i)]^{\frac{1}{3}} \qquad (3.3)$$

4. RELATION BETWEEN CUBIC AND ARITHMETIC EXPECTATIONS

Arithmetic expectation [7 , 9] of random variable X, denoted by $E_A(X)$, is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i$$

when X is finite

and by

$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i$$

when X is countable infinite (denumerable)

and also, by

$$E_A(X) = \int_a^b x.f(x)dx$$

when X is continuous.

Similarly, the arithmetic expectation of a function φ (X) of X, denoted by $E_A\{\varphi(X)\}$, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i x_i$$

when X is finite

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i x_i$$

when X is countable infinite (denumerable)

and also, by

$$E_A\{\varphi(X)\} = \int_a^b x.f(x)dx$$

when X is continuous.

Putting

$$\varphi(X) = X^3$$

in any of the three definitions of $EA\{\varphi(X)\}$ and in any of the three definitions of $EC\{\varphi(X)\}$, it is obtained that

$$\{E_C(X)\}^3 = E_A(X^3) \quad \text{or} \quad E_C(X) = \{EA(X^3)\}^{\frac{1}{3}} \qquad (4.1)$$

Thus, cubic expectation of a random variable X is the cube root of arithmetic expectation of its cube (i.e. X^3).

Similarly, putting

$$\varphi(X) = \{h(X)\}^3$$

in any of the three definitions of $EA\{\varphi(X)\}$ and in any of the three definitions of $EC\{\varphi(X)\}$, it is obtained that

$$[EC\{h(X)\}]^3 = EA[\{h(X)\}]^3 \quad \text{or} \quad EC\{h(X)\} = EA[\{h(X)\}]^{\frac{1}{3}} \qquad (4.2)$$

$$\text{i.e.} \quad [EC\{\varphi(X)\}]^3 = EA[\{\varphi(X)\}]^3 \quad \text{or} \quad EC\{\varphi(X)\} = EA[\{\varphi(X)\}]^{\frac{1}{3}} \qquad (4.3)$$

i.e. the cubic expectation of a function $\varphi(X)$ of X is the cube root of arithmetic expectation of the cube of $\varphi(X)$ i.e. of $\{\phi(X)\}^3$.

Thus, the following relationship, mentioned as theorem, between cubic expectation and arithmetic expectation has been obtained:

Theorem (4.1):

"The cubic expectation of a random variable X is the cube root of the arithmetic expectation of its cube i.e. of X^3 and also the cubic expectation of a function $\varphi(X)$ of a random variable X is the cube root of the arithmetic expectation of the cube of $\varphi(X)$ i.e. of $\{\phi(X)\}^3$."

4.1 Numerical Example

Let us consider the example of probability distribution of number of rainy days at New Delhi in the month June, shown in Table – 4.1, discussed in an earlier study [7 , 8] where arithmetic expectation, geometric expectation, harmonic expectation and quadratic expectation of number of rainy days in the month June at New Delhi were found to be

4.53125 , 4.0548509573441987687123105790392 , 3.4579759862778730703259005145797

& 4.9212549212573818873031465305075

respectively.

Table -4.1:

June	
Number/Interval of Rainy Days	**Probability of occurrence**
1	0.0625
2	0.09385
[3 , 5]	0.5625

[6 , 8]	0.25
9	0.03125
> 9	0

The values of $p_i x_i 3$ have been found as follows:

Table -4.2:

Number/Interval of Rainy Days	Mid value (x_i)	Value of $p_i x_i 3$
1	1	0.0625
2	2	0.75
3 – 5	4	36
6 – 8	7	85.75
9	9	22.78125
Total	23	145.34375

Applying the above formula, the cubic expectation of number of rainy days in the month June at New Delhi has been found to be

5.2577361382748444625896679373977

5. CONCLUSION

The cubic mean (or root mean cube) is a type of mathematical average that is particularly important for analysing data related to volume, 3D structures, and physical processes where values increase proportional to the cube of a dimension [21].

Cubic mean {also known as root mean) is a significant measure of average due to its importance and applications in various fields like

Biology (Morphometric): where it is used to measure the mean dimensions of spherical bacteria and other approximately spheroidal organisms,

Engineering (Life Expectancy of Parts): where it is used in predicting the life expectancy of machine parts,

Wind Energy Analysis: where it is used as a measure of local potential for wind energy,

Physical Sciences: where it is useful in applications involving 3D modelling and calculations of volumetric data

and many other [2 , 6 , 14 , 19, 20 , 21].

Consequently, cubic expectation like arithmetic expectation, geometric expectation, harmonic expectation & quadratic expectation has become a significant concept in analysis of data. It is to be noted that expectation is a theoretical entity whose value is usually unknown and is to be estimated from data available in sample. One criterion/quality of a good estimate is unbiasedness which is defined on the basis of expectation of estimator [16]. Therefore, cubic expectation can lead to make a concept of cubic unbiasedness of estimator and to define it by the similar logic which was used in making concepts as well as in defining of arithmetic unbiasedness, geometric unbiasedness, harmonic unbiasedness & quadratic unbiasedness of estimator [10 , 11 , 13].

REFERENCES

[1] Anders Hald (1990): "History of Probability and Statistics and Their Applications before 1750", Wiley Series in Probability and Statistics, ISBN (Print): 9780471502302 ISBN (Online): 9780471725169. DOI:10.1002/0471725161.

[2] Babu & Sridhar (2010): "Design of Machine Elements" McGraw-Hill Education (India) Pvt Limited. ISBN 9780070672840.

[3] Chai T. (2022): "Root Mean Square", In: Daya Sagar, B.S., Cheng, Q., McKinley, J., Agterberg, F. (eds) Encyclopedia of Mathematical Geosciences. Encyclopedia of Earth Sciences Series. Springer, Cham. https://doi.org/10.1007/978-3-030-26050-7_280-1.

[4] Chattamvelli R., Shanmugam R. (2024):"Mathematical Expectation", In: Random Variables for Scientists and Engineers, Synthesis Lectures on Engineering, Science, and Technology. Springer, Cham. https://doi.org/10.1007/978-3-031-58931-7_1.

[5] Coggeshall F. (1886): "The Arithmetic, Geometric, and Harmonic Means"", The Quarterly Journal of Economics, 1(1), 83-86.

[6] Da Rosa Aldo Vieira (2013): "*Fundamentals of Renewable Energy Processes*", Academic Press, Page 696. ISBN 9780123972194.

[7] Dhritikesh Chakrabarty (2024): "Idea of Arithmetic, Geometric and Harmonic Expectations", Partners Universal International Innovation Journal (PUIIJ, 02(01), 119 – 124. www.puiij.com . DOI:10.5281/zenodo.10680751.

[8] Dhritikesh Chakrabarty (2025): "Quadratic Expectation and Some of Its Properties", Partners Universal Innovative Research Publication (PUIRP), 03(02), 74 – 79. www.puirp.com. 10.5281/zenodo.15292622.

[9] Dhritikesh Chakrabarty (2025): "Continuous Random Variable Arithmetic, Geometric, Harmonic and Quadratic Expectations", Partners Universal Innovative Research Publication (PUIRP), (ISSN: 3048-586X), 03(04), 36 – 42. www.puirp.com. DOI:10.5281/zenodo.16996732.

[10] Dhritikesh Chakrabarty (2025): "Geometric Unbiased Estimator: Some Properties", Partners Universal International Innovation Journal (PUIIJ), 03(05), 69 – 75. www.puiij.com. DOI:10.5281/zenodo.17444460.

[11] Dhritikesh Chakrabarty (2025): "Harmonic Unbiased Estimator: Some Properties", Partners Universal Innovative Research Publication (PUIRP), 03(05), 92 – 96. www.puirp.com. DOI:10.5281/zenodo.17536909.

[12] Dhritikesh Chakrabarty (2025): "Cubic Mean of Cubic Means of Possible Subsets of a Set of Numbers", International Journal of Advanced Research in Science, Engineering and Technology, 12(11), 23988 – 23996. www.ijarset.com.

[13] Dhritikesh Chakrabarty (2025): "Quadratic Unbiased Estimator: Some Properties", Partners Universal Innovative Research Publication (PUIRP), 03(06), 136 – 142. www.puirp.com. DOI:10.5281/zenodo.18037586.

[14] Harris Tedric A. & Kotzalas Michael N. (2006): "Essential Concepts of Bearing Technology Fifth Edition". CRC Press. ISBN 9781420006599.

[15] HELM (2008): "The Mean Value and the Root-Mean-Square Value", Workbook 14, Section 14.2, 10 – 19. https://www.sheffield.ac.uk › media › download.

[16] Lehmann E. L. (1951): "A General Concept of Unbiasedness", *The Annals of Mathematical Statistics*, 22(4), 587 – *592.* doi.10.1214/aoms/1177729549. JSTOR 2236928.

[17] Pavel Polasek (1979): "The Significance of the Root Mean Square Velocity Gradient and Its Calculation in Devices for Water Treatment", Water SA, 5(4), 196 – 207.

[18] Pfeiffer P.E. (1990): "Mathematical Expectation", In: Probability for Applications. Springer Texts in Statistics, Springer, New York, NY. https://doi.org/10.1007/978-1-4615-7676-1_15.

[19] Rice Dale W. & Wolman Allen A. (1971): "The Life History and Ecology of the Gray Whale (Eschrichtius Robustus)", Stillwater, Oklahoma: American Society of Mammalogists. Page 34.

[20] Rodina Antonina Gavrilovna (1972): "*Methods in Aquatic Microbiology*", University Pvark Press, Page 158. ISBN 083910071X.

[21] Svarovski Ladislav (2000): "Solid-Liquid Separation. Elsevier. ISBN 9780080541440.

[22] Yadav S. K., Singh S., Gupta R. (2019): "Random Variable and Mathematical Expectation", In: Biomedical Statistics, Springer, Singapore. https://doi.org/10.1007/978-981-32-9294-9_26.

[23] Weisstein Eric W.: "Arithmetic Mean", From MathWorld--A Wolfram Resource. https://mathworld.wolfram.com/ArithmeticMean.html.

[24] Weisstein Eric W.: "Geometric Mean", From MathWorld--A Wolfram Resource. https://mathworld.wolfram.com/GeometricMean.html.

[25] Weisstein Eric W.: "Harmonic Mean", From MathWorld--A Wolfram Resource. https://mathworld.wolfram.com/HarmonicMean.html.

[26] Weisstein Eric W.: "Root-Mean Mean", From MathWorld--A Wolfram Resource. https://mathworld.wolfram.com/Root-Mean-Square.html.

[27] Whittle P. (2000): "Probability via Expectation, 4th ed., Springer, https://link.springer.com/book/10.1007/978-1-4612-0509-8.