

## Quadratic Unbiased Estimator: Some Properties

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**Abstract** – In an earlier study, concept of quadratic unbiased estimator was introduced and defined on the basis of quadratic expectation. Attempt has here been made to identify some important properties of quadratic unbiased estimator. This article is based on the information obtained in the attempt, on the properties of this unbiased estimator.

**Keywords:** Estimation, Quadratic Unbiasedness, QUE, Some Properties.

### 1. INTRODUCTION

Unbiasedness, in the literature of statistical estimation, is regarded as a desirable property/quality/criterion of an estimator [1, 12]. Originally, the concept of unbiasedness [12] was explained on the basis of the mathematical expectation [2, 14, 17], more specifically the arithmetic expectation [3, 6], of the estimator concerned and accordingly unbiased estimator was defined [15, 16]. This definition later was termed as arithmetic unbiased estimator [5]. In some studies, as continuation to arithmetic unbiased estimator, concepts of geometric unbiased estimator [5], harmonic unbiased estimator [5] & quadratic unbiased estimator [5] had been introduced and defined on the basis of geometric expectation [3, 6], harmonic expectation [3, 6] & quadratic expectation [4, 6] respectively along with identifying some properties of geometric unbiased estimator [7] & of harmonic unbiased estimator [8]. Here attempt has here been made on identifying some important properties of quadratic unbiased estimator. This article is based on the information on this unbiased estimator obtained in the attempt.

### 2. QUADRATIC AND ARITHMETIC UNBIASED ESTIMATORS

Suppose,

$$X_1, X_2, \dots, X_n$$

is a real valued random sample drawn from a population of a non-zero real valued random variable  $X$  which follows a probability distribution having real valued parameter  $\theta$

$$\& \quad T = T(X_1, X_2, \dots, X_n)$$

is an estimator of  $\theta$ .

Then  $T$  is quadratic unbiased estimator of parameter  $\theta$  if

$$E_Q(T) = \theta$$

where  $E_Q(T)$  is the quadratic expectation of  $T$ .

Similarly, if

$$S = S(X_1, X_2, \dots, X_n)$$

is an estimator of  $\phi(\theta)$ , a function of parameter  $\theta$ ,

then  $S$  is quadratic unbiased estimator of  $\phi(\theta)$  if

$$E_Q(S) = \phi(\theta)$$

$T$  is arithmetic unbiased estimator of parameter  $\theta$  if

$$E_A(T) = \theta$$

where  $E_A(T)$  is the arithmetic expectation of  $T$ .

and  $S$  is arithmetic unbiased estimator of  $\phi(\theta)$  if

$$E_A(S) = \phi(\theta)$$

Let us abbreviate quadratic unbiased estimator by QUE and arithmetic unbiased estimator by AUE.

### 3. SOME ALGEBRAIC PROPERTIES

Property (1): If  $T$  is QUE of  $\theta$  the  $T^2$  is AUE of  $\theta^2$  and if  $T$  is AUE of  $\theta$  the  $T^{\frac{1}{2}}$  is QUE of  $\theta^{\frac{1}{2}}$ .

Proof: Quadratic expectation of a random variable  $X$  is the absolute square root of arithmetic expectation of its square [4, 6] i.e.

$$E_Q(X) = \{E_A(X^2)\}^{\frac{1}{2}}$$

where  $E_A(X)$  is the arithmetic expectation of  $X$ .

This implies,

$$E_Q(X^{\frac{1}{2}}) = \{E_A(X)\}^{\frac{1}{2}}$$

Thus,

$$E_Q(T) = \{E_A(T^2)\}^{\frac{1}{2}} \ \& \ E_Q(T^{\frac{1}{2}}) = \{E_A(T)\}^{\frac{1}{2}}$$

The first one of these two implies,

$$\text{if } E_Q(T) = \theta \text{ then } E_A(T^2) = \theta^2$$

i.e. if  $T$  is QUE of  $\theta$  the  $T^2$  is AUE of  $\theta^2$ .

Similarly the second one of these two implies,

$$\text{if } E_A(T) = \theta \text{ then } E_Q(T^{\frac{1}{2}}) = \theta^{\frac{1}{2}}$$

i.e. if  $T$  is AUE of  $\theta$  the  $T^{\frac{1}{2}}$  is QUE of  $\theta^{\frac{1}{2}}$ .

Property (1): If  $T$  is QUE of parameter  $\theta$  then  $cT$  is QUE of parameter  $c\theta$  where  $c$  is non-zero real constant.

Proof:  $E_Q(cT)$  can be expressed as

$$E_Q(cT) = [E_A\{(cT)^2\}]^{\frac{1}{2}} = \{c^2 E_A(T^2)\}^{\frac{1}{2}} = c\{E_A(T^2)\}^{\frac{1}{2}} = c.E_Q(T) = c\theta$$

Therefore,  $cT$  is QUE of parameter  $c\theta$ .

Corollary: If  $T$  is QUE of parameter  $\theta$  then  $-T$  is QUE of parameter  $-\theta$ .

This follows from the fact that

$$E_Q(-T) E_Q(-1.T) = -1.E_Q(T) = -\theta$$

Property (2): Quadratic mean (abbreviated as QM) of a finite number of QUEs of a parameter  $\theta$  is also QUE of the parameter  $\theta$ .

Proof: Suppose, T and S are two QUEs of a parameter  $\theta$ .

Equation

$$E_Q(X) = \{E_A(X^2)\}^{\frac{1}{2}}$$

implies that

$$E_Q(X^{\frac{1}{2}}) = \{E_A(X)\}^{\frac{1}{2}}$$

Thus,

$$\begin{aligned} EQ(\text{QM of T and S}) &= E_Q\left[\left\{\frac{1}{2}(T^2 + S^2)\right\}^{\frac{1}{2}}\right] = [E_A\left\{\frac{1}{2}(T^2 + S^2)\right\}]^{\frac{1}{2}} \\ &= \left[\frac{1}{2}\{E_A(T^2) + E_A(S^2)\}\right]^{\frac{1}{2}} \end{aligned}$$

But

$$E_A(T^2) = E_Q(T)^2 = \theta^2 \quad \& \quad E_A(S^2) = \{E_Q(S)\}^2 = \theta^2$$

Therefore,

$$E_Q(\text{QM of T and S}) = \left\{\frac{1}{2}(\theta^2 + \theta^2)\right\}^{\frac{1}{2}} = \theta$$

Therefore, QM of T and S is QUE of  $\theta$ .

Now suppose,

$$T_1, T_2, \dots, T_r$$

are r QUEs of a parameter  $\theta$ .

Then

$$\begin{aligned} &E_Q(\text{QM of } T_1, T_2, \dots, T_r) \\ &= E_Q\left[\left\{\frac{1}{r}(T_1^2 + T_2^2 + \dots + T_r^2)\right\}^{\frac{1}{2}}\right] \\ &= [E_A\left\{\frac{1}{r}(T_1^2 + T_2^2 + \dots + T_r^2)\right\}]^{\frac{1}{2}} \\ &= \left[\frac{1}{r}\{E_A(T_1^2) + E_A(T_2^2) + \dots + E_A(T_r^2)\}\right]^{\frac{1}{2}} \end{aligned}$$

But

$$E_A(T_1^2) = E_Q(T_1)^2 = \theta^2,$$

$$E_A(T_2^2) = E_Q(T_2)^2 = \theta^2,$$

..... ,

$$E_A(T_r^2) = E_Q(T_r)^2 = \theta^2$$

Therefore,

$$E_Q (\text{QM of } T_1, T_2, \dots, T_r) = \left\{ \frac{1}{r} (\theta^2 + \theta^2 + \dots + \theta^2) \right\}^{\frac{1}{2}} = \theta$$

Therefore, QM of  $T_1, T_2, \dots, T_r$  is QUE of  $\theta$ .

Property (3): There may exist more than one QUE of a parameter.

Proof: Let us consider a population following the uniform discrete distribution [13] described by the probability mass function

$$P(X = x_i) = \frac{1}{K}, \quad (x_i = 1, 2, \dots, K)$$

with population QM  $\mu_Q$  where

$$\mu_Q = \left( 1 \cdot \frac{1}{K} + 2 \cdot \frac{1}{K} + \dots + K \cdot \frac{1}{K} \right)^{\frac{1}{2}}$$

Suppose,

$$\{X_1, X_2, \dots, X_n\}$$

is a random sample drawn from this population.

Then each element of the sample assumes the values

1, 2, ..., k

with equal probability  $\frac{1}{K}$ ,

so that by the definition of quadratic expectation,

$$E_Q (X_i) = \left( 1 \cdot \frac{1}{K} + 2 \cdot \frac{1}{K} + \dots + K \cdot \frac{1}{K} \right)^{\frac{1}{2}} = \mu_Q,$$

for each  $X_i$  ( $i = 1, 2, \dots, k$ ).

This implies each  $X_i$  is a QUE of  $\mu_Q$ .

By Property (2),

QM of any two elements of the sample is QUE of  $\mu_Q$ ,

Similarly, QM of any three elements of the sample is also QUE of  $\mu_Q$ ,

QM of any four elements of the sample is also QUE of  $\mu_Q$

and so on.

Thus, a parameter may have more than one QUE of itself.

Property (4): There may not exist QUE of a parameter.

Proof: Let us consider a population following Poisson distribution [9, 10, 11] having parameters  $\lambda$  described by the probability mass function

$$P(X = x) = \frac{\exp(-\lambda)}{x!} \cdot \lambda^x, \quad (x = 1, 2, 3, \dots)$$

Suppose

$$X_1, X_2, \dots, X_n$$

is a random sample drawn from this population.

For this distribution, QUE of the binomial parameter  $\lambda$  does not exist.

Thus, Property (4) has been established.

Property (5): If T is QUE of parameter  $\theta$  and S is QUE of parameter  $\varphi$  then QM of T and S is QUE of the QM of  $\theta$  and  $\varphi$ .

In general, if

$$T_1, T_2, \dots, T_r$$

are r QUEs of the respective parameters

$$\theta_1, \theta_2, \dots, \theta_r,$$

then the QM of  $T_1, T_2, \dots, T_r$ , is QUE of the QM of  $\theta_1, \theta_2, \dots, \theta_r$ .

Proof: We have

$$\begin{aligned} E_Q(\text{QM of T and S}) &= E_Q\left[\left\{\frac{1}{2}(T_2 + S_2)\right\}^{\frac{1}{2}}\right] = \left[E_A\left\{\frac{1}{2}(T_2 + S_2)\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{2}\{E_A(T_2) + E_A(S_2)\}\right]^{\frac{1}{2}} \end{aligned}$$

But

$$E_A(T_2) = \{E_Q(T)\}^2 = \theta^2 \quad \& \quad E_A(S_2) = \{E_Q(S)\}^2 = \varphi^2$$

Therefore,

$$E_Q(\text{QM of T and S}) = \left\{\frac{1}{2}(\theta^2 + \varphi^2)\right\}^{\frac{1}{2}} = \text{QM of } \theta \text{ and } \varphi$$

Therefore, QM of T and S is QUE of QM of  $\theta$  and  $\varphi$ .

Again,

$$T_1, T_2, \dots, T_r$$

are r QUEs of the respective parameters

$$\theta_1, \theta_2, \dots, \theta_r,$$

This implies,

$$E_A(T_{1,2}) = \{E_Q(T_1)\}^2 = \theta_1^2,$$

$$E_A(T_{2,2}) = \{E_Q(T_2)\}^2 = \theta_2^2,$$

..... ,

$$E_A(T_{r,2}) = \{E_Q(T_r)\}^2 = \theta_r^2$$

This implies,

$$\begin{aligned} & E_Q (\text{QM of } T_1, T_2, \dots, T_r) \\ &= E_Q \left[ \left\{ \frac{1}{r} (T_1^2 + T_2^2 + \dots + T_r^2) \right\}^{\frac{1}{2}} \right] \\ &= \left[ E_A \left\{ \frac{1}{r} (T_1^2 + T_2^2 + \dots + T_r^2) \right\} \right]^{\frac{1}{2}} \\ &= \left[ \frac{1}{r} \{ E_A(T_1^2) + E_A(T_2^2) + \dots + E_A(T_r^2) \} \right]^{\frac{1}{2}} \\ &= \left\{ \frac{1}{r} (\theta_1^2 + \theta_2^2 + \dots + \theta_r^2) \right\}^{\frac{1}{2}} \\ &= \text{QM of } \theta_1, \theta_2, \dots, \theta_r \end{aligned}$$

Therefore,

$$\text{QM of } T_1, T_2, \dots, T_r$$

is QUE of the

$$\text{QM of } \theta_1, \theta_2, \dots, \theta_r.$$

#### 4. CONCLUSION

It is hoped that the concept of quadratic unbiasedness may be useful and/or helpful in finding unbiased estimator of a parameter in the situation where the associated data are such that their arithmetic mean cannot be calculated but the arithmetic mean of their squares can be calculated. Also, the properties of quadratic unbiased estimator, obtained here, are likely to be helpful in finding unbiased estimator of a function of parameter in similar situations. On the whole, it can be concluded that the properties of quadratic unbiased estimator, mentioned above, can contribute to making the theory of statistical estimation more developed.

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