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Harmonic Unbiased Estimator: Some Properties

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Abstract - In an earlier study, concept of harmonic unbiased estimator was introduced and defined based on harmonic expectation. Attempt has here been made to identify some important properties/facts/results of harmonic unbiased estimator. This article is based on the information on this unbiased estimator obtained in the attempt.

Keywords: Estimation, Harmonic Unbiasedness, HUE, Some Properties.

1. INTRODUCTION

Unbiasedness, in the literature of statistical estimation, is regarded as a desirable property/quality/criterion of an estimator [1 , 8 , 12, 13]. Originally, the concept of unbiasedness [8 , 9] was explained on the basis of the mathematical expectation [2 , 11 , 14], more specifically the arithmetic expectation, of the estimator concerned and accordingly unbiased estimator was defined [1 , 12 , 13]. This definition later was termed as arithmetic unbiased estimator [6]. Recently, concepts of harmonic unbiased estimator [6] was introduced and defined based on harmonic expectation [3 , 4 , 5 , 7]. Attempt has here been made on identifying some important properties/facts/results of harmonic unbiased estimator. This article is based on the information on this unbiased estimator obtained in the attempt.

2. HARMONIC UNBIASED ESTIMATOR

Suppose,

$$X_1 , X_2 , \ldots\ldots , X_n$$

is a random sample drawn from a population of a non-zero real valued random variable X which follows a probability distribution having parameter θ

& $T = T(X_1 , X_2 , \ldots\ldots , X_n)$

is an estimator of θ.

Then T can be regarded as harmonic unbiased estimator of parameter θ if

$$E_H(T) = \theta$$

where $E_H(T)$ is the harmonic expectation of T.

Let us abbreviate harmonic unbiased estimator by HUE.

Note:

HUE exists in the case of non-zero real valued estimator. The corresponding parameter θ, in this case, is an unknown non-zero real number.

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3. ALGEBRAIC PROPERTIES OF HARMONIC UNBIASED IN ESTIMATOR

Property (1): If T is HUE of parameter θ then c.T is HUE of parameter c.θ.

Proof: This follows from the fact that

$$E_H(c.T) = c.E_H(T) = c.\theta$$

Corollary: If T is HUE of parameter θ then -T is HUE of parameter -θ.

This follows from the fact that

$$E_H(-T)\ E_H(-1.T) = = -1.E_H(T) = -\theta$$

Property (2): Harmonic mean (HM) of a finite number of HUEs of a parameter θ is also HUE of the parameter θ.

Proof: Suppose, T and S are two HUEs of a parameter θ.

Then

E_H (HM of T and S) = $E_H\left(\frac{2}{\frac{1}{T}+\frac{1}{S}}\right)$

$= 2E_H\{(\frac{1}{T}+\frac{1}{S})^{-1}\}$

$= 2\{EA(\frac{1}{T}+\frac{1}{S})\}^{-1}$, (where EA(T) is the arithmetic expectation of T)

$= 2\{E_A(\frac{1}{T})+ E_A(\frac{1}{S})\}^{-1}$

$= 2[\{E_H(T)\}^{-1}+ \{E_H(S)\}^{-1}]^{-1}$

$= 2(\theta^{-1}+\theta^{-1})^{-1}$

$= \theta$

Therefore, HM of T and S is HUE of a parameter θ.

Now suppose,

$T_1\ , T_2\ ,\ \ldots\ldots\ldots\ ,\ T_r$

are r HUEs of a parameter θ.

Then

HM of $T_1\ , T_2\ ,\ \ldots\ldots\ldots\ ,\ T_r = \frac{r}{\frac{1}{T_1}+\frac{1}{T_2}+\cdots\ldots\ldots+\frac{1}{T_r}}$

Proceeding similarly as in the earlier case, one can obtain that

$E_H\left(\frac{r}{\frac{1}{T_1}+\frac{1}{T_2}+\cdots\ldots\ldots+\frac{1}{T_r}}\right) = \theta$

Therefore, HM of $T_1\ , T_2\ ,\ \ldots\ldots\ldots\ ,\ T_r$ is HUE of θ.

Thus, Property (2) has been proved for a finite number of estimators.

Hence, Property (2) has been established.

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Property (3): There may exists more than one HUE of a parameter.

Proof: Let us consider a population following the uniform discrete distribution [10] described by the probability mass function

$$P(X = x_i) = \frac{1}{K} \quad , \quad (x_i = 1, 2, \ldots\ldots, K)$$

with population harmonic mean μH where

$$\mu_H = \left(\frac{1}{\frac{1}{K}\sum_{i=1}^{n}\frac{1}{K}}\right)$$

Suppose,

$$X_1, X_2, \ldots\ldots, X_n$$

is a random sample drawn from this population.

Then each element of the sample assumes the values

1 , 2 , ... , k

with equal probability $\frac{1}{K}$,

so that by the definition of harmonic expectation,

$E_H(X_i) = \left(\frac{1}{\frac{1}{K}\sum_{i=1}^{n}\frac{1}{K}}\right) = \mu_H$, for each X_i $(i = 1, 2, \ldots\ldots, k)$

This implies each X_i is a HUE of μ_H .

By Property (3),

HM of any two elements of the sample is HUE of μ_H .

Similarly, HM of any three elements of the sample is also HUE of μ_H ,

HM of any four elements of the sample is also HUE of μ_H

and so on.

Thus, Property (3) has been established.

Property (4): There may not exists HUE of a parameter.

Proof: Let us consider a population following binomial distribution [10] having parameters R (number of trials) and p (probability of success).

Suppose

$$X_1, X_2, \ldots\ldots, X_n$$

is a random sample drawn from this population.

For this distribution, HUE of the binomial parameter p does not exist.

Thus, Property (4) has been established.

Property (5): If T is HUE of parameter θ and S is HUE of parameter φ then HM of T and S is HUE of the HM of θ and φ.

In general, if

$$T_1 \ , T_2 \ , \ .... \ , \ T_r$$

are r HUEs of the respective parameters

$$\theta_1 \ , \theta_2 \ , \ .... \ , \theta_r . ,$$

then the HM of $T_1 \ , T_2 \ , \ .... \ , \ T_r$, is HUE of the HM of $\theta_1 \ , \theta_2 \ , \ .... \ , \theta_r$.

Proof: This follows from the fact that We

E_H (HM of T and S) = $E_H \left(\frac{2}{\frac{1}{T}+\frac{1}{S}}\right)$

= $2E_H \{(\frac{1}{T}+\frac{1}{S})^{-1}\}$

= $2\{E_A(\frac{1}{T}+\frac{1}{S})\}^{-1}$, (where $E_A(T)$ is the arithmetic expectation of T)

= $2\{E_A(\frac{1}{T})+ E_A(\frac{1}{S})\}^{-1}$

= $2[\{E_H(T)\}^{-1}+ \{E_H(S)\}^{-1}]^{-1}$

= $2(\theta^{-1}+ \varphi^{-1})^{-1}$

= HM HM of θ and φ

Proceeding similarly as in the earlier case, one can obtain that

E_H (HM of $T_1 \ , T_2 \ , \ .... \ , \ T_r$) = HM of $\theta_1 \ , \theta_2 \ , \ .... \ , \theta_r$

Hence, Property (5) has been established.

4. CONCLUSION

Concept of geometric unbiasedness is likely to be useful and/or helpful in finding unbiased estimator of a parameter in the situation where the associated data are of ratio type or other allied types.

The properties of geometric unbiased estimator, obtained here, are likely to be useful and/or helpful in finding unbiased estimator of a function of parameter in the similar situations.

Moreover, the properties of geometric unbiased estimator are likely to be important and useful in enriching the theory of statistical estimation.

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