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Continuous Random Variable Arithmetic, Geometric, Harmonic and Quadratic Expectations

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Abstract - Concept of geometric, harmonic, and quadratic expectations had been introduced and defined in the case of discrete random variables in some recent studies. The concepts and definitions of these three types of expectations have here been extended to the case of continuous random variable. Descriptions of these, with example, have been presented in this article.

Keywords: Random Variable, Continuous, Expectation, Geometric, Harmonic, Quadratic.

1. INTRODUCTION

Expectation (also termed as expected value, expectancy) of a random variable is a theoretical concept associated and is defined as the weighted average of its all possible values with their respective probabilities as the corresponding weights [1 , 2 , 7 , 8]. It was defined first with the help of arithmetic mean [3 , 5 , 19 , 23 , 25] and the definition was termed as mathematical expectation [1 , 2 , 6 , 7 , 14 , 15 , 20 , 21 , 24]. Though, this definition of expectation is popularly known as mathematical expectation, it is more specific to term it as arithmetic expectation [8].

In some other studies, expectation had been defined with the help of geometric mean [3 , 5 , 16 , 19 , 22 , 23 , 26], harmonic mean [3 , 5 , 17 , 19 , 23 , 27] and quadratic mean (also termed as root mean square) [4 , 18 , 23 , 28] and the definitions were termed as geometric expectation [8 , 9 , 10], harmonic expectation [8 , 9 , 11 , 12] and quadratic expectation respectively [13]. In those studies, concept of geometric, harmonic & quadratic expectations had been introduced and defined in the case of discrete random variables. The concepts and definitions of these three types of expectations have here been extended to the case of continuous random variable. Descriptions of these, with example, have been presented in this article

2. FOUR TYPES OF EXPECTATION FOR CONTINUOUS RANDOM VARIABLE

Let us first logically proceed with defining arithmetic expectation of a continuous random variable. Geometric expectation, harmonic expectation, and quadratic expectation can be defined by the same logic.

2.1. Arithmetic Expectation

If X is a discrete random variable assuming the values

$$x_1 , x_2 , \ldots\ldots\ldots , x_n$$

with respective probabilities

$$p_1 , p_2 , \ldots\ldots\ldots , p_n$$

then arithmetic expectation of X denoted by EA(X) is defined by

$$EA(X) = \sum_{i=1}^{n} p_i \, x_i$$

Now, suppose that X, instead of discrete, is a continuous random variable assuming values in the real valued interval

(a , b) or [a , b) or (a , b] or [a , b]

with probability density function f(x).

Since the possible values in an interval is uncountable & infinite, summation of these values is to be obtained by integration.

Accordingly, EA(X) in this case is to be defined by

$$EA(X) = \int_a^b x.f(x)dx$$

It is to be noted that

(1) EA(X) is defined for any random variable assuming real values.

& (2) in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

each a & b of may be finite or infinite.

Therefore, arithmetic expectation of a continuous random variable can be defined as follows:

Definition (1):

If X is a continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then arithmetic expectation of X denoted by EA(X) can be defined by

$$EA(X) = \int_a^b x.f(x)dx$$

2.2. Geometric Expectation

For existence of geometric expectation EG(X) of the discrete random variable X, it is to be noted that the values

$$x_1 , x_2 , \text{..........} , x_n$$

Assumed by X are to be all positive real.

EG(X) in this case is defined by

$$EG(X) = \text{antilog} \{\sum_{i=1}^{n} p_i \log x_i\} = \exp \{ \sum_{i=1}^{n} p_i \log e \, xi \}$$

By the same logic as in the case of arithmetic expectation, when X is a continuous random variable assuming positive real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

EG(X) becomes

$$EG(X) = \exp\left\{\int_a^b \log_e x . f(x) dx\right\}$$

Here, it is to be noted that

(1) EG(X) is defined for any random variable assuming positive real values.

& (2) in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

each a & b of may be finite or infinite but must be positive.

Therefore, geometric expectation of a continuous random variable can be defined as follows:

Definition (2):

If X is a continuous random variable assuming positive real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be but must be positive,

having probability density function f(x),

then geometric expectation of X denoted by EG(X) can be defined by

$$EG(X) = \exp\left\{\int_a^b \log_e x . f(x) dx\right\}$$

2.3. Harmonic Expectation

For existence of harmonic expectation EH(X) of the discrete random variable X, it is to be noted that the values

$$x_1 , x_2 , \ldots\ldots\ldots , x_n$$

assumed by X are to be all non-zero real.

EH(X) in this case is defined by

$$EH(X) = \frac{1}{\sum_{i=1}^{n} p_i \frac{1}{x_i}}$$

By the same logic as in the case of arithmetic expectation, when X is a continuous random variable assuming non-zero real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

EH(X) becomes

$$EH(X) = \frac{1}{\int_a^b \frac{1}{x}.f(x)dx}$$

Here, it is to be noted that

(1) EG(X) is defined for any random variable assuming non-zero real values.

& (2) in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

each a & b of may be finite or infinite but must be non-zero.

Therefore, harmonic expectation of a continuous random variable can be defined as follows:

Definition (3):

If X is a continuous random variable assuming non-zero real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be but must be non-zero,

having probability density function f(x),

then harmonic expectation of X denoted by EH(X) can be defined by

$$EH(X) = \frac{1}{\int_a^b \frac{1}{x}.f(x)dx}$$

2.4. Quadratic Expectation

For existence of quadratic expectation EQ(X) of the discrete random variable X, it is to be noted that the values

$$x_1 , x_2 , \dots\dots\dots , x_n$$

assumed by X are to be all real.

EQ(X) in this case is defined by

$$EQ(X) = (\sum_{i=1}^{n} p_i x_i 2)^{\frac{1}{2}}$$

By the same logic as in the case of arithmetic expectation, when X is a continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

EQ(X) becomes

$$EQ(X) = \sqrt{\int_a^b x^2 .f(x)dx}$$

Here, it is to be noted that

(1) EG(X) is defined for any random variable assuming real values.

& (2) in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

each a & b of may be finite or infinite.

Therefore, quadratic expectation of a continuous random variable can be defined as follows:

Definition (4):

If X is a continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

where a , b may be finite or infinite,

having probability density function f(x),

then quadratic expectation of *X* denoted by EQ(X) can be defined by

$$EQ(X) = \sqrt{\int_a^b x^2 . f(x) dx}$$

3. EXAMPLE

Suppose a continuous random variable X follows the uniform distribution whose probability density function is described by

$$f(x) = k \quad , c < x < d \, , \; c < d \, , \; c > 0 \; , \; d > 0$$

Here, first it is required to express the constant k in terms of c & d.

The property of the probability density function namely

$$\int_c^d k.f(x)dx = 1 \quad \text{i.e.} \quad \int_c^d k.f(x)dx = 1$$

implies that

$$k = \frac{1}{d-c}$$

which further implies that

$$f(x) = \frac{1}{d-c} \quad , c < d \, , \; c > 0 \; , \; d > 0$$

Since X assumes positive real values therefore all of arithmetic expectation, geometric expectation, harmonic expectation & quadratic expectation exist.

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Now

EA(X) = $\int_c^d x.(\frac{1}{d-c})dx$ = $\frac{d+c}{2}$,

EG(X) = $\exp\{\int_c^d \log_e x.(\frac{1}{d-c})dx\}$ = $\exp(\frac{d\log_e d - c\log_e c}{d-c} - 1)$,

EH(X) = $\frac{1}{\int_c^d \frac{1}{x}.(\frac{1}{d-c})dx}$ = $\frac{d-c}{\log_e d - \log_e c}$ = $\frac{d-c}{\log_e(\frac{d}{c})}$

& EQ(X) = $\sqrt{\int_c^d x^2.(\frac{1}{d-c})dx}$ = $\sqrt{\frac{c^2+cd+d^2}{3}}$.

4. CONCLUSION

In this article, the definitions of arithmetic, geometric, expectation & quadratic expectations that were developed in the case of discrete random variable in some earlier studies have been extended to the case of continuous random variable. The definitions obtained in this study can be in dealing with the data continuous in nature.

In earlier studies, some properties of each of arithmetic expectation, geometric expectation, harmonic expectation & quadratic expectation were established mathematically. However, those properties were mathematically proved for discrete random variable. It is also to know whether the properties, established for discrete random variable in those studies, hold good in the case of continuous random variable.

It is, in this connection, to be mentioned that the definitions of arithmetic, geometric, expectation & quadratic expectations have, till this date, been developed in the case of discrete random variable as well as in the case of continuous random variable. Definitions of them are yet to be extended/developed denumerable (i.e. countable infinite) random variable.

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