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Quadratic Expectation and Some of Its Properties

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Abstract – Expectation of a random variable, which had already been defined with the help of arithmetic mean, geometric mean & harmonic mean, has here been defined on the basis of quadratic mean (also called root mean square) and termed as quadratic expectation since it is based on quadratic mean. This article describes this measure of expectation numerical examples.

Keywords: Random Variable, Expectation, Measure, Quadratic Mean.

1.INTRODUCTION

The idea/concept of the expected value originated in the middle of the 17th century from the study of the so-called problem of points, which seeks to divide the stakes in a fair way between two players, who have to end their game before it is properly finished [1]. This problem had been debated for centuries. Many conflicting proposals and solutions had been suggested over the years when it was posed to Blaise Pascal by French writer and amateur mathematician Chevalier de Méré in 1654 who also discussed the problem in the famous series of letters to Pierre de Fermat. Soon enough they both independently came up with solutions/results which were identical though solved in different computational ways but they did not publish their findings while they only informed a small circle of mutual scientific friends in Paris about it [20].

In Dutch mathematician **Christiaan Huygens'** book published his treatise in 1657 [15 , 16], he considered the problem of points, and presented a solution based on the same principle as the solutions of Pascal and Fermat. Extended concept of expectation was developed/described in the book, by adding rules for how to calculate expectations in more complicated situations than the original problem, which and could be seen as the first successful attempt at laying down the foundations of the **theory of probability**. Neither Pascal nor Huygens used the term "expectation" in its modern sense. [15 , 16]. More than a hundred years later, in 1814, **Pierre-Simon Laplace** published his tract "Théorie analytique des probabilités", where the concept of expected value was defined explicitly [18].

In the mid-nineteenth century, Pafnuty Chebyshev became the first person to think systematically in terms of the expectations of random variables [13 , 28] while W. A. Whitworth, in 1901, started the use of the letter E to denote "expected value" [29].

In mathematics/statistics, a variable is defined as an entity which assumes many (more than one) certain values while a random variable is defined as an entity which assumes many (more than one) uncertain/possible values [26]. Consequently, expected value or simply expectation of a random variable is defined as weighted average of its all possible values with their respective probabilities as corresponding weights [3 , 22 , 23 , 28]. Arithmetic expectation (commonly termed as mathematical expectation in standard literature of statistics), is the oldest and widely used measure of expectation [3 , 8 , 9 , 22 , 23 , 28]. This measure of expectation had been defined with the help of arithmetic mean [5 , 27]. Moreover, two other measures of expectation had also been defined on the basis of geometric mean [5] and harmonic mean [5 , 26] which were respectively termed as geometric expectation [8, 9 , 10] and harmonic expectation [8 , 9 , 11 , 12]. Since it had been found possible to measure expectation of a random variable with the help of the arithmetic mean, geometric mean & harmonic mean which are three measures of average [6 , 7], it can also be possible to define the same with the help of other measures of average. That is why the same has here been defined on the basis of quadratic mean (also called root mean square) [2 , 14 , 17 , 25]. It has been termed as quadratic expectation since it is based on quadratic mean. This article describes this measure of expectation numerical examples.

2. QUADRATIC EXPECTATION

Let us consider a discrete finite random variable.

Suppose, X assumes the values

$$x_1 , x_2 , \dots\dots\dots , x_N$$

These N values constitute the population of the variable X.

Q(X) = Q($x_1 , x_2 , \dots\dots\dots , x_N$), the quadratic mean [2 , 25] of X, is defined by

$$Q(X) = \{ \tfrac{1}{N} (x_1 2 + x_2 2 + \dots\dots\dots + x_N 2)\}^{\frac{1}{2}} = (\textstyle\sum_{i=1}^{N} \ x_N 2)^{\frac{1}{2}} \qquad (2.1)$$

On the other hand, if the values of X correspond to the respective weights

$$w_1 , w_2 , \dots\dots\dots , w_N$$

then, Q(X) = Q($x_1 , x_2 , \dots\dots\dots , x_N$), will be weighted quadratic mean of X and is accordingly defined by

$$Q(X) = (\tfrac{1}{\sum_{i=1}^{N} w_i} \textstyle\sum_{i=1}^{N} \ w_i x_i)^{\frac{1}{2}} \qquad (2.2)$$

Now suppose, X assumes the values

$$x_1 , x_2 , \dots\dots\dots , x_N$$

with respective probabilities

$$p_1 , p_2 , \dots\dots\dots , p_N$$

Then Q(X), in this case, will be the weighted quadratic mean of X with the probabilities as the weights of its respective possible values and accordingly is defined by

$$Q(X) = (\textstyle\sum_{i=1}^{N} \ p_i x_i 2)^{\frac{1}{2}} \quad , \text{ since } \textstyle\sum_{i=1}^{N} p_i = 1 \qquad (2.3)$$

Since it describes the weighted quadratic mean of all possible values of X with respective probabilities as the weights of the respective values, it can be regarded as the quadratic expectation of X. Thus quadratic expectation can be defined as follows:

Definition:

If a random variable X assumes the values $x_1 , x_2 , \dots\dots\dots , x_N$ with respective probabilities $p_1 , p_2 , \dots\dots\dots , p_N$, then the quadratic expectation of X, denoted by EQ(X), is defined by

$$EQ(X) = (\textstyle\sum_{i=1}^{N} \ p_i x_i 2)^{\frac{1}{2}} \qquad (2.3)$$

3. SOME BASIC PROPERTIES

Property (1): Quadratic mean is a measure of average which measures the absolute magnitude of a set of numbers. Accordingly, quadratic expectation describes the theoretical absolute magnitude of a random variable.

Property (2): Arithmetic expectation of the random variable, denoted by EA(X), is defined by

$$EA(X) = \sum_{i=1}^{N} p_i x_i$$

which implies $EA(X2) = \sum_{i=1}^{N} p_i x_i 2$

i.e. $\{ EQ(X) \}^2 = EA(X\,2)$ or $EQ(X) = \{ EA(X\,2) \}^{\frac{1}{2}}$ (3.1)

Thus, quadratic expectation of a random variable X can also be defined as the absolute square root of arithmetic expectation of its square (i.e. X 2).

Property (3): From **Property (2),** it is obtained that

$$EQ\,(\sqrt{X}\,) = \sqrt{\{EA\,(X\,)\}}$$

Additive property of arithmetic expectation [3 , 10 – 12 , 22 , 23] implies that X_k

$$EA\,(X+Y) = EA\,(X) + EA\,(Y)$$

which together with the earlier implies that

$$\{EQ\,(\sqrt{X+Y}\,)\}^2 = \{EQ\,(\sqrt{X}\,)\}^2 + \{EQ\,(\sqrt{Y}\,)\}^2 \qquad (3.2)$$

This can be regarded as additive property of quadratic expectation for two random variables.

Similarly, for k random variables

$$X_1\,, X_2\,, \,\ldots\ldots\ldots\,, X_k\,,$$

it can be obtained that

$$\{EQ\,(\sqrt{X_1 + X_2 + \ldots\ldots\ldots + X_k}\,)\}^2 = \{EQ\,(\sqrt{X_1}\,)\}^2 + \{EQ\,(\sqrt{X_2}\,)\}^2 + \ldots\ldots$$
$$\ldots\ldots + \{EQ\,(\sqrt{X_k}\,)\}^2 \qquad (3.3)$$

This can be regarded as general additive property of quadratic expectation.

Property (4):

(4a) From the inequality

$\{ EA(X2) – EA(X) \}^2 > 0$, since square of an expression is > 0

it is obtained that

$$EA(X2) > \{EA(X)\}^2 \quad \text{i.e.} \quad \{ EA(X2) \}^{\frac{1}{2}} > EA(X)$$

Hence, $EQ(X) > EA(X)$ (3.4)

(4b) It was found in a study that arithmetic expectation EA(X), geometric expectation EG(X) & harmonic expectation EH(X) satisfies the inequality [8]

$$EA(X) > EG(X) > EH(X)$$

Hence, the following inequality is obtained:

$$EQ(X) > EA(X) > EG(X) > EH(X) \qquad (3.5)$$

Equality holds good when X comes down to be a constant.

4. NUMERICAL EXAMPLE

Let us consider the example of probability distribution of number of rainy days at New Delhi in the month June, shown in Table – 4.1, discussed in an earlier study [8] where arithmetic expectation, geometric expectation and harmonic expectation of number of rainy days in the month June at New Delhi were found to be 4.53125 , 4.0548509573441987687123105790392 & 3.4579759862778730703259005145797 respectively.

Table -4.1:

June	
Number/Interval of Rainy Days	**Probability of occurrence**
1	0.0625
2	0.09385
[3 , 5]	0.5625
[6 , 8]	0.25
9	0.03125
> 9	0

In order to find out the values of quadratic expectation, we are to construct the following table (Table – 4.2):

Table -4.2:

Number/Interval of Rainy Days	Mid value (x_i)	Probability of occurrence (p_i)	Value of $p_i x_i 2$
1	1	0.0625	0.0625
2	2	0.09375	0.375
3 – 5	4	0.5625	9.0
6 – 8	7	0.25	12.25
9	9	0.03125	2.53125
Total		1.0	24.21875

Applying the above formula, the quadratic expectation of number of rainy days in the month June at New Delhi has been found to be

4.92125492125738188730314653O5075

5. CONCLUSION

The quadratic mean, is calculated by squaring each value, averaging the squares, and then taking the square root of that average, is a type of measure of average which emphasizes gives more weight to larger values because of the squaring operation.

Due to the importance of quadratic mean in various fields like

(1) Physics, Engineering & Signal Processing to measure effective magnitude or average value, particularly when dealing with fluctuating quantities,

(2) Statistics for measuring the deviation or variance of a dataset from its mean,

(3) Error Analysis to determine the root mean square error, which is a common metric for evaluating the performance of a model or algorithm,

(4) Forestry to characterize the group of trees in a forest, assigning greater weight to larger trees,

etc. [4 , 13 , 19 , 21 , 24], quadratic expectation may be a significant theoretical tool of analyzing phenomena in these fields.

It is to be noted that quadratic mean focuses on the effective magnitude or "strength" of a set of values, while arithmetic mean focuses on the algebraic value.

Accordingly, quadratic expectation focuses on theoretical effective magnitude or "strength" of a set of values, while the arithmetic mean expectation on the theoretical algebraic value.

Expectation is a theoretical concept, associated to a random variable, which in general is measures/defined by weighted average of all the possible values assumed by the variable weighted by the respective probabilities of the possible values assumed by it. Quadratic expectation is a special measure/definition of expectation where quadratic mean is used as the measure of average.

It is to be mentioned that the four measures/definitions of expectation among which the existing three are arithmetic expectation, geometric expectation & harmonic expectation and the 4th one is quadratic expectation that has been defined here may not suit all types of data set since types of data in different situations are different. Accordingly, there is necessity of developing more measures/definitions of expectation.

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